SOLUTION OF TWO-DIMENSIONAL HEAT-CONDUCTION PROBLEMS

BY THE Z TRANSFORM METHOD

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The Z-transform (discrete Laplace transform) is used to solve heat-conduction problems in axisymmetric bodies of arbitrary shape for different types of boundary conditions.

The mathematical apparatus of the Z-transform (or discrete Laplace transform) is widely used in the dynamics of control systems [1, 2]. The Z-transform is also effective in the case when the control system is the temperature field of a body, for example an element of a power plant [3]. In this case the problem reduces to the solution of the heat-conduction equation by the Z-transform method. This method has been used in the solution of one-dimensional problems for the field in a bar and in a cylinder [3, 4]. Below we consider the case of a two-dimensional axisymmetric body of arbitrary shape. Many of the parts of steam and gas turbines, and other power equipment, such as rotors, regulating valves, etc., can be reduced to this form. As an example, consider a body formed by rotation about the x axis of a contour whose boundary (Fig. 1) is arbitrarily divided into two parts: Γ^* represents the union of segments 0-1-2-3-4-5, and Γ^{**} represents segments 5-6-7-8-9-0.

The geometry shown in Fig. 1 corresponds to regulating and cut-off valves in turbines, and is shown only as an illustration; the computations done below will be for very different systems.

For simplicity we will only consider cases where the heat flux vanishes on the portion of the boundary Γ^{**}:

$$(r^{**},\ x^{**}) \in \Gamma^{**},\ \left.\frac{\partial t}{\partial N}\right|_{\Gamma^{**}} = 0. \tag{1}$$

External effects on the temperature field of the body occur through the surface Γ^*.

The heat-conduction equation for the body has the form

$$\frac{1}{a}\,\frac{\partial t(\tau,\ r,\ x)}{\partial \tau} = \frac{\partial^2 t}{\partial r^2} + \frac{1}{r}\,\frac{\partial t}{\partial r} + \frac{\partial^2 t}{\partial x^2} = \nabla^2 t(\tau,\ r,\ x). \tag{2}$$

We use the concept of a grid function of time $t[n\Delta\tau,\ r,\ x]$, or in abbreviated notation $t[n,\ r,\ x]$, whose values are defined at the discrete times $\tau = n\Delta\tau$. Values of the grid functions $t[n,\ r,\ x]$ are identical to those of the continuous function $t(\tau,\ r,\ x)$ at the same instants of time; the function $t(\tau,\ r,\ x)$ can be thought of as enveloping the grid function $t[n,\ r,\ x]$. The analog of the first derivative of a continuous function is the first (backward) difference of the grid function:

$$\Delta t[n,\ r,\ x] = t[n,\ r,\ x] - t[n-1,\ r,\ x]. \tag{3}$$

With the help of (3) and after transformation to relative coordinates, the heat-conduction equation (2) takes the form

$$\nabla^2 t[n,\ \rho,\ u] - f^{-1}\Delta t[n,\ \rho,\ u] = 0. \tag{4}$$

We take the Z-transform of this equation with the help of the relation [1]

$$Z\{t[n]\} = T(z,\ \rho,\ u) = \sum_{n=0}^{\infty} t[n]\,z^{-n}, \tag{5}$$

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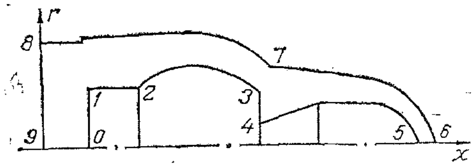

Fig. 1. Geometry for the boundary conditions on the surface of a body of revolution.

and this brings (4) to the form

$$\nabla^2 T(z,\ \rho,\ u) - f^{-1}(1-z^{-1})\,T(z,\ \rho,\ u) = 0. \tag{6}$$

Where we use some properties of the discrete Z-transform [1]: multiplication of T(z) by z^{-1} corresponds to a time lag of one discrete step, i.e., $Z^{-1}\{z^{-1}T(z)\} = t[n-1]$; if the grid function identically vanishes for negative values of its argument, then the transform of the k-th backward difference is given by

$$Z\{\Delta^k t[n]\} = (1-z^{-1})^k. \tag{5a}$$

We consider boundary conditions of types I and II on the surface Γ*. It can be shown that the solution for type III boundary conditions is a combination of the solutions for type I and II boundary conditions.

Type I boundary conditions are represented in the form

$$(\rho^*,\ u^*) \in \Gamma^*,\ t^{\mathrm{I}}[n,\ \Gamma^*] = \sum_{j=1}^{k} t_j^{\mathrm{I}}[n,\ \Gamma^*],\ t_j^{\mathrm{I}}[n,\ \Gamma^*] = y_j^{\mathrm{I}}[n]\,\varphi_j^{\mathrm{I}}(\Gamma^*). \tag{7}$$

Type II boundary conditions can be written in similar form

$$\frac{\partial t^{\mathrm{II}}[n,\ \Gamma^*]}{\partial N} = \sum_{j=1}^{k} \frac{\partial t_j^{\mathrm{II}}[n,\ \Gamma^*]}{\partial N},\quad \frac{\partial t_j^{\mathrm{II}}[n,\ \Gamma^*]}{\partial N} = y_j^{\mathrm{II}}[n]\,\varphi_j^{\mathrm{II}}(\Gamma^*). \tag{8}$$

The function $y_j^l(\tau)$ or $y_j^l[n]$ (l = I, II) are known arbitrary functions of time, describing external effects on the temperature field of the body.

Obviously, the general solution of the problem for the temperature field of the body will be a superposition of solutions, corresponding to each of the functions $y_j^l[n]$:

$$T^l[n,\ \rho,\ u] = \sum_{j=1}^{k} t_j^l[n,\ \rho,\ u]. \tag{9}$$

Application of the Z-transform to (7) and (8) leads to:

$$T_j^{\mathrm{I}}(z,\ \Gamma^*) = Y_j^{\mathrm{I}}(z)\,\varphi_j^{\mathrm{I}}(\Gamma^*), \tag{7a}$$

$$\frac{\partial T_j^{\mathrm{II}}(z,\ \Gamma^*)}{\partial N} = Y_j^{\mathrm{II}}(z)\,\varphi_j^{\mathrm{II}}(\Gamma^*). \tag{8a}$$

We seek the solution of (6) for the j-th component of the field in a form similar to that in the case of a one-dimensional field [4]:

$$T_j^l(z,\ \rho,\ u) = Y_j^l(z) \sum_{i=0}^{\infty} R_{ji}^l(\rho,\ u)\, f^{-i}(1-z^{-1})^i. \tag{10}$$

Substitution of (10) into (6) and taking into account the arbitrary nature of the functions $y_j^l[n]$, results in a recursive set of differential equations determining the functions $R_{ji}^l(\rho,\ u)$:

$$l = \mathrm{I},\ \mathrm{II},\ \nabla^2 R_{ji}^l(\rho,\ u) = R_{j,i-1}^l(\rho,\ u),\ i \geqslant 1,\ \nabla^2 R_{j0}^l. \tag{11}$$

The boundary conditions for the R_{ji}^l are determined by substitution of (10) into (1), (7), and (8):

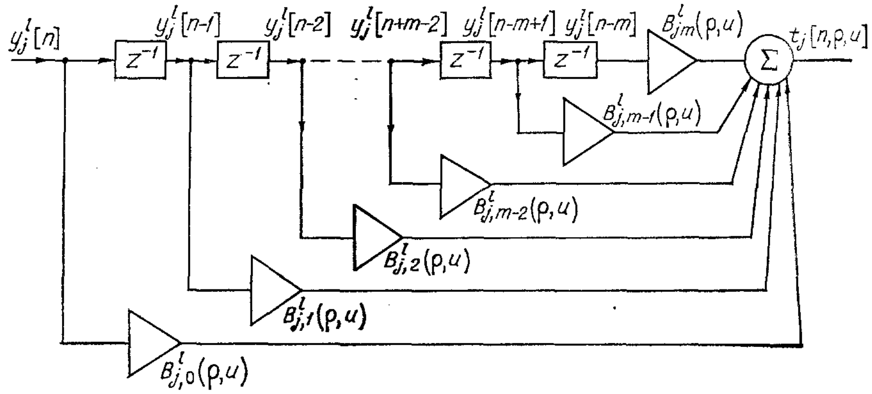

Fig. 2. Block diagram of a computational algorithm (nonrecursive filter) corresponding to (16).

$$\frac{\partial R_{ji}^l(\Gamma^{**})}{\partial N}=0,\ i=0,\ 1,\ 2,\ \ldots,\ j=1,\ 2,\ 3,\ \ldots,\ k,\ l=\mathrm{I},\ \mathrm{II}, \tag{12a}$$

$$l=\mathrm{I},\ R_{j0}^{\mathrm{I}}(\Gamma^*)=\varphi_j^{\mathrm{I}}(\Gamma^*),\ R_{ji}^{\mathrm{I}}(\Gamma^*)=0,\ i\geqslant 1, \tag{12b}$$

$$l=\mathrm{II},\ \frac{\partial R_{j0}^{\mathrm{II}}(\Gamma^*)}{\partial N}=\varphi_j^{\mathrm{II}}(\Gamma^*),\quad \frac{\partial R_{j0}^{\mathrm{II}}(\Gamma^*)}{\partial N}=0,\ i>1. \tag{12c}$$

Performing the inverse transform with the help of (5a), we obtain

$$t^l[n,\ \rho,\ u]=\sum_{j=1}^{k} t_j^l[n,\ \rho,\ u],\ t_j^l[n,\ \rho,\ u]=\sum_{i=0}^{\infty}\Delta^i y_j^l[n]\,R_{ji}^l(\rho,\ u)\,f^{-i}. \tag{13}$$

If we use the relation

$$\Delta^i y^l[n]=\sum_{\nu=0}^{i}(-1)^\nu C_i^\nu y^l[n-\nu], \tag{14}$$

where the C_i^ν are the binomial coefficients (number of combinations) and introduce the notation

$$B_{j\nu}^l(\rho,\ u)=(-1)^\nu\sum_{i=0}^{n} C_i^\nu R_{j\nu}^l(\rho,\ u)\,f^{-i}, \tag{15}$$

then

$$t_j^l[n,\ \rho,\ u]=\sum_{\nu=0}^{n} B_{j\nu}^l(\rho,\ u)\,y_j^l[n-\nu]. \tag{16a}$$

In terms of the transform functions, (16a) has the form

$$T_j^l(z,\ \rho,\ u)=Y_j^l(z)\sum_{\nu=0}^{n} B_{j\nu}^l(\rho,\ u)\,z^{-\nu}. \tag{16b}$$

The computational scheme given by (16a), (16b) can be expressed in terms of so-called nonrecursive filters [2]. A block diagram of the simplest nonrecursive filter is shown in Fig. 2. It consists of elements, corresponding to the operations of summation, amplification (i.e., multiplication by a constant) and time-lag. The diagram corresponds to a limited number $m \leqslant n$ of terms in the sum in (16). For engineering purposes, analysis shows that it is sufficient to limit the number of terms to m = 3-5.

For a discrete model of a heated body, nonrecursive filters are not always the most effective, and in some respects are inferior to recursive filters.

Recursive filters [2, 4] allow one to obtain more information on the temperature field for the same number of elements (elementary computational operations), and also lead to a better convergence to the exact solution for the same value of m. Transformation of (10) and (16) to expressions corresponding to a recursive filter can be done in the following manner.

As an external function we take the temperature at one of the inner points of the body with coordinates (ρ_A, u_A):

$$V_j^l(z) = T_j^l(z, \rho_A, u_A), \ l = \mathrm{I}, \mathrm{II}. \tag{17}$$

We then obtain the solution in the form

$$T_j^l(z, \rho, u) = V_j^l(z) \sum_{i=0}^{\infty} P_{ji}^l(\rho, u) f^{-i}(1 - z^{-1})^i. \tag{18}$$

It can be shown that the accuracy of the method, and in particular the number of terms in the series under the summation sign in the solution necessary to obtain the required accuracy depend on choice of the point A. Without dwelling in detail on this question, we show that in the absence of additional complications, point A must be chosen such that the thermal impedance between it and neighboring points on the surface subject to external effects be a maximum [5]. At a point within the cross section of the body, the function $R_{j0}(\rho)$ is a minimum, while the $R_{ji}(\rho)$, $i \geqslant 1$ take maximum (absolute) values.

At point (ρ_A, u_A) the functions P_{ji}^l take the following values

$$l = \mathrm{I}, \mathrm{II}, \ P_{j0}^l(\rho_A, u_A) = 1, \ P_{j\nu}^l(\rho_A, u_A) = 0, \ \nu = 1, 2 \ldots \tag{19}$$

At other points, the P_{ji}^l are determined through the functions $R_{j\nu}^l(\rho, u)$, by expressing the external functions from (10) in terms of $V_j^l(z)$; after substitution of the result again in (10) we have

$$T_j^l(z, \rho, u) = V_j^l(z) \frac{\sum_{i=0}^{\infty} R_{ji}^l(\rho, u) f^{-i}(1 - z^{-1})^i}{\sum_{i=0}^{\infty} R_{ji}^l(\rho_A, u_A) f^{-i}(1 - z^{-1})^i}. \tag{20}$$

Equating (18) and (20) gives

$$P_{ji}^l(\rho, u) = \frac{1}{R_{j0}^l(\rho_A, u_A)} \left[R_{ji}^l(\rho, u) - \sum_{\nu=1}^{i} R_{j\nu}(\rho_A, u_A) P_{j,i-\nu}(\rho, u) \right]. \tag{21}$$

The functions $P_{ji}^l(\rho, u)$ can be determined directly without prior calculation of the $R_{ji}^l(\rho, u)$, It can be shown that these functions are solutions of the recursive set of equations

$$l = \mathrm{I}, \mathrm{II}, \ \nabla^2 P_{ji}^l(\rho, u) = P_{j,i-1}^l(\rho, u), \ i \geqslant 1, \ \nabla^2 R_{j0}^l = 0 \tag{22}$$

subject to the boundary conditions

$$\frac{\partial P_{ji}^l(\Gamma^{**})}{\partial N} = 0, \ i = 0, 1, \ldots, \ j = 1, 2, 3, \ldots, k, \ l = \mathrm{I}, \mathrm{II}, \tag{23}$$

$$l = \mathrm{I}, \ \frac{1}{\varphi_j^{\mathrm{I}}(\Gamma^*)} P_{j\nu}^{\mathrm{I}}(\Gamma^*) = \mathrm{idem}, \ l = \mathrm{II}, \ \frac{1}{\varphi_j^{\mathrm{II}}} \frac{\partial P_{j\nu}^{\mathrm{II}}(\Gamma^*)}{\partial N} = \mathrm{idem} \tag{24}$$

and condition (19).

We transform (18) to a form similar to (16):

$$T_j^l(z, \rho, u) = V_j^l(z) \sum_{\nu=0}^{n} A_{j\nu}^l(\rho, u) z^{-\nu}. \tag{25}$$

Expressing the function $V_j^l(z)$ in terms of the original external function $Y_j^l(z)$, we obtain

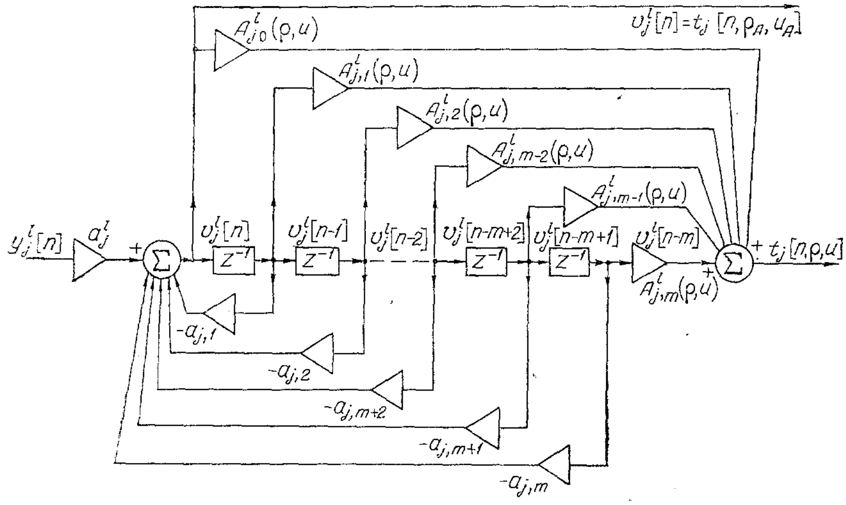

Fig. 3. Block diagram of a computational algorithm (recursive digital filter) corresponding to (27).

$$T^l_j(z,\ \rho,\ u) = Y^l_j(z)\,\frac{\sum\limits_{\nu=0}^{n} A^l_{j\nu}(\rho,\ u)\,z^{-\nu}}{\sum\limits_{\nu=0}^{n} A^l_{j\nu}(\Gamma^{**})\,z^{-\nu}}, \tag{26}$$

which is really two equations; one of them is (25), the other

$$Y^l_j(z) = V^l_j(z)\sum_{\nu=0}^{n} A^l_{j\nu}(\Gamma^*)\,z^{-\nu}.$$

We now take the inverse transform to get

$$t^l_j(z,\ \rho,\ u) = \sum_{0}^{n} v^l_j[n-\nu]\,A^l_{j\nu}(\rho,\ u), \tag{27a}$$

$$v^l_j[n] = a^l_{j0}\,y^l_j[n] - \sum_{1}^{n} a^l_{jn}\,y^l_j[n-\nu], \tag{27b}$$

where $a^l_{j0} = 1/A^l_{0j}(\Gamma^*)$; $a^l_{j\nu} = A^l_{\nu j}(\Gamma^*)/A^l_{0j}(\Gamma^*)$.

The result (26) and relations (27a) and (27b) deduced from it represent to a computational scheme corresponding to a recursive filter as shown in Fig. 3. Other filters can be obtained with application of the techniques described in [2, 4].

Application of the above solution is particularly effective when the number k of external functions y_j = (j = 1, 2, 3,..., k) is small. In particular, values k = 1, 2, 3 are typical in practical problems of computer control of transient thermal states of bodies. On the other hand, computers normally used for this purpose have a limited operational memory and this determines the severity of the decrease in the volume of computational operations carried out in real time.

Because the adjusted parameters (for example, parameters characterizing the reliability of the process of heating the body) are usually related to values of the temperature at a limited number of points on the body, application of our method of solution satisfies this requirement. Values of the functions $P_{j\nu}(\rho,\ u)$ or $R_{j\nu}(\rho,\ u)$, within required accuracy, are determined from solution of the appropriate equations using large or medium computers by grid methods or Green's function methods. Only the values of these functions at actual points in the body are used; they enter as coefficients in expressions which correspond to computer control processes. In particular, the author has used solutions obtained by the methods discussed here to build control algorithms for heating processes in high-power steam turbines at start-up.

NOTATION

t, temperature; r, x, geometrical coordinates; L, characteristic size of the body; τ, time; $\Delta\tau$, discrete time step. l = I, II; n, m, k, integers; $\Delta\tau^* = L^2/a$; $\rho = r/\sqrt{a\Delta\tau^*}$; $u = x/\sqrt{a\Delta\tau^*}$; a, diffusivity; s, Laplace transform variable; ∇^2, Laplacian operator; Γ, boundary of the body; y, v, functions characterizing the external effect on the temperature field; $T(z)$, $Y(z)$, $V(z)$, Z-transforms of the functions t, y, v; $z = \exp(s\Delta\tau)$; $f = \Delta\tau/\Delta\tau^*$.

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